MATH LEAGUE: Formulas, Facts and Tips

PYTHAGOREAN TRIPLES

3, 4, 5	7, 24, 25	12, 35, 37
5, 10, 12	9, 40, 41	20, 21, 29
8, 15, 17	11, 60, 61	

and their multiples

PYTHAGOREAN THM (RT.Δ'S)

$$a^2 + b^2 = c^2$$

FACTORING

Difference of cubes:

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

Sum of cubes:

$$a^3 + b^3 = (a+b)(a^2-ab+b^2)$$

Factoring by parts: Group and factor groups, then factor results.

$$ab + ac + bd + cd = a(b+c) + d(b+c) = (a+d)(b+c)$$

QUADRATIC FORMULA:

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum of the roots $= -b/a$

Product of the roots $= c/a$

Axis of Symmetry of $y = ax^2 + bx + c$ is $x = -b/2a$

DIVISIBILITY RULES:

by 2 : last digit is even

by 3 : sum of the digits is div. by 3

by 4: last 2 digits are divisible by 4

by 5: last digit is 5 or 0

by 10: last digit is 0

by 8: last 3 digits are divisible by 8

by 11: in two digit numbers, digits are the same
in three digit numbers, the ones and hundreds digit adds to the tens

FRACTIONS

F.F.F.: Factor Fractions First

"Poly wanna Factor" : Polynomials should be in factored form before cancelling or determining a L.C.D

To rationalize binomial denominators, multiply by the conjugate form of 1:

$$\frac{a}{b-\sqrt{c}} \bullet \frac{b+\sqrt{c}}{b+\sqrt{c}}$$

POWERS OF i

$i = \sqrt{-1} = i^5$

$i^2 = -1 = i^6$

$i^3 = -i = i^7$

$i^4 = +1 = i^8$

Continues in cycles of 4

PERMUTATIONS

$${}_nP_r = \frac{n!}{(n-r)!}$$

Arrangements

COMBINATIONS

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Selections

BINOMIAL EXPANSION

$$(a+b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \dots + {}_nC_{n-1}a^1b^{n-1} + {}_nC_na^0b^n$$

ARITHMETIC PROGRESSIONS

Sum; $S = \frac{n}{2}(a + l)$

a= 1st term, l = last term, n = # terms, d= common difference: $l = a + (n-1)d$

PASCAL'S Δ

Sum in each row is 2^n

Powers of 11

11^0 =	1		${}_0C_0$
11^1 =	1 1		${}_1C_0$ ${}_1C_1$
11^2 =	1 2 1		${}_2C_0$ ${}_2C_1$ ${}_2C_2$
11^3 =	1 3 3 1	≡	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
11^4 =	1 4 6 4 1		${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
Etc	1 5 10 10 5 1		etc.
	1 6 15…etc.		${}_nC_k + {}_nC_{k+1} = {}_{n+1}C_{k+1}$

GEOMETRIC PROGRESSIONS

r = common ratio

geom. Mean = $\sqrt{xy}$

$$l = a \bullet r^{n-1}$$

$$S = \frac{a - rl}{1 - r} = \frac{a - ar^n}{1 - r}$$

SUM OF INFINITE GEOM. SERIES

$$S = \frac{a}{1 - r}$$

LOG RULES

Conversion Rules

To base 10:

$$\log_B A = \frac{\log A}{\log B}$$

To any base:

$$\log_B A = \frac{1}{\log_A B} \qquad \log_B N = \frac{\log_A N}{\log_A B}$$

Rewriting rules:

$$\log_b(m \cdot n) = \log_b m + \log_b n \qquad \log_b m^p = p \cdot \log_b m$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n \qquad \log_b \sqrt[p]{m} = \frac{1}{p} \cdot \log_b m$$

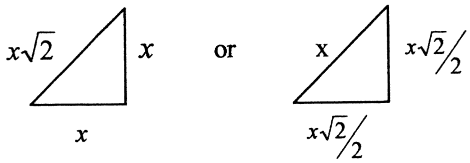
NUMBERS FROM THEIR DIGITS:

h t u

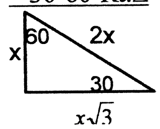
Number = 100h + 10t + u Number reversed = 100u + 10t + h

SPECIAL RT ΔS:

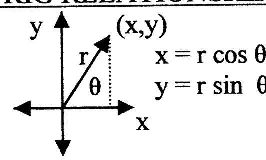
Isosceles Δ Rt



30-60-Rt.Δ



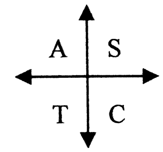
TRIG RELATIONSHIPS:



x = r cos θ

y = r sin θ

π radians = 180°



sin 90° = 1

sin 30° = 1/2

sin 45° = $\sqrt{2}/2$

sin 60° = $\sqrt{3}/2$

Identities:

$\sin^2\theta + \cos^2\theta = 1$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Negative Angles

sin (-A) = - sin A

cos (-A) = cos A

LAW OF COSINES:

$a^2 = b^2 + c^2 - 2bc\cos A$

LAW OF SINES:

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

AREA OF Δ

$A = \dfrac{1}{2}ab\sin C$

SUMS OF ANGLES:

$\sin(A + B) = \sin A\cos B + \sin B\cos A$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

DOUBLES OF ANGLES:

$\sin 2A = 2\sin A\cos A$ $\qquad \cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$ $\qquad \cos 2A = \cos^2 A - \sin^2 A$

HALF ANGLE:

$\sin\left(A/2\right) = \pm\sqrt{\dfrac{1 - \cos A}{2}}$

OBSCURE TRIG LAWS:

$\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$ $\qquad \cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

DE MOIVRE'S THM:

Any imaginary number ***a + bi*** can be represented by ***r*(cos *x* + *i* sin *x*)**, abbreviated ***rcisx.***

$$(rcisx)^n = r^n[cis(nx)]$$

GEOMETRY FORMULAS & FACTS:

Equilateral Δ

$A = \dfrac{s^2}{4}\sqrt{3}$

Rhombus

$A = \dfrac{1}{2}d_1 \cdot d_2$

Trapezoid

$A = \dfrac{1}{2}h(b_1 + b_2)$

Heron's Formula for Area of Δ

$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{1}{2}(a + b + c)$

Circle

$C = 2\pi r = \pi d$

$A = \pi r^2$

Cylinder

$V = \pi r^2 h$

$SA = 2\pi r^2 + 2\pi rh$

Cube

$V = s^3$

$SA = 6s^2$

Rectangular Prism

$V = LWH$ $\qquad$ Diagonal = $\sqrt{L^2 + W^2 + H^2}$

SA = sum of area of 6 faces

Sphere

$V = \dfrac{4}{3}\pi r^3$

$SA = 4\pi r^2$

Pyramid

$V = \dfrac{1}{3}Bh$

(*B* is area of base)

Cone

$V = \dfrac{1}{3}\pi r^2 h$

$SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$

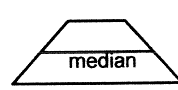
Point-Slope Eq. of Line:

$y - y_1 = m(x - x_1)$

Obscure Thm: The number of distinct regions of a circle with an inscribed polygon whose diagonals are not concurrent is:

$${}_nC_4 + {}_nC_2 + 1$$

Median of a Trapezoid:



The median in a trapezoid is the line segment connecting the midpoints of the non-parallel sides.

$m = \frac{1}{2}(b_1 + b_2)$

Formulas, Facts and Tips(cont'd)

Polygons:

Regular: equilateral and equiangular
Convex: all interior angles less than 180°
Concave: at least one reflex angle > 180°

The **apothem** of a regular polygon is the radius of it's inscribed circle, and is the ⊥ bisector of the polygon's side.

Area of polygon = ½ *ap*

where a is apothem, p is perimeter

Sum of int. angles of n-agon: 180(n – 2)
Exterior angle of n-agon: 360/n
Number of diagonals in n-agon: $\frac{n(n-3)}{2}$

In Similar Polygons

ratio of sides = ratio of perimeters
ratio of areas = (ratio of sides)^2
ratio of volumes = (ratio of sides)^3

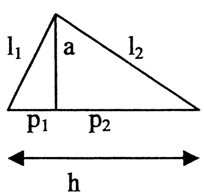
Def :

3or more lines are **concurrent** if they intersect in one point.
centroid: the intersection of the 3 medians.
circumcenter: the intersection of the 3 ⊥ bisectors of the sides.
incenter: the intersection point of the 3 angle bisectors.

GEOMETRY FACTS AND FORMULAS

Altitude to hypotenuse in rt Δ:

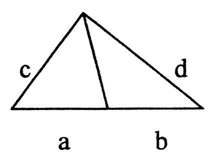
forms 3 ~ Δs



$\frac{p_1}{a} = \frac{a}{p_2}$ or $a^2 = (p_1)(p_2)$

$\frac{l_1}{h} = \frac{p_1}{l_1}$ or $(l_1)^2 = h(p_1)$

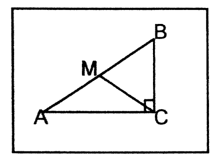
Angle Bisector in a Δ



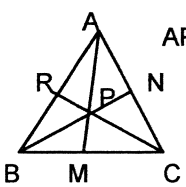
$\frac{c}{a} = \frac{d}{b}$

Median to hypotenuse in Rt Δ

CM = ½ AB



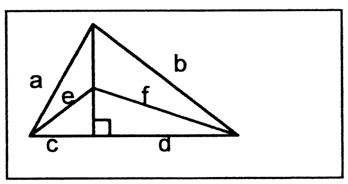
Medians in a Δ .



AP =(2/3)AM
AP = 2(PM)
BP = 2(PN)
CP = 2(PR)

Medians divide Δ into 6 Δs = in area

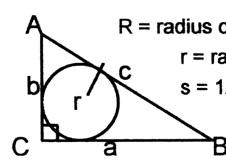
Common Altitudes



$a^2 - c^2 = b^2 - d^2$

$a^2 - e^2 = b^2 - f^2$

Rt. Δs and Circles

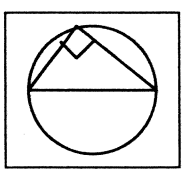


R = radius of circumscribed circle
r = radius of inscribed circle
s = 1/2 (a+b+c)
Area = rs

$r = \frac{a+b-c}{2}$ $Area = \frac{(abc)}{4R}$

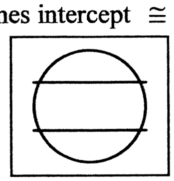
Circles

Circles : Central < = *m* intercepted arc
Inscribed < = ½ m intercepted arc

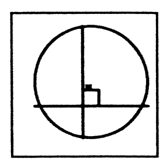


A Δ inscribed in a semicircle is a rt. Δ

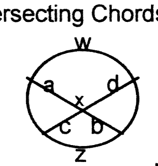
≅ chords intercept ≅ arcs
‖ lines intercept ≅ arcs



radius or diameter ⊥ to a chord, bisects the chord



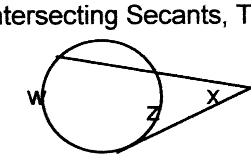
Intersecting Chords



segments:
ab = cd

angles:
(vertex inside circle)
m<x = 1/2(w+z)

Intersecting Secants, Tangents



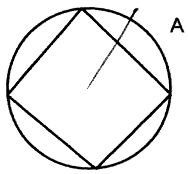
Segments:
(whole segment)(its external piece)=(whole segment)(external piece)

Angles:
(vertex outside the circle)
m< x = ½ (w – z)

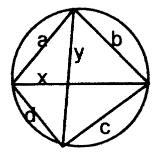
Tangents to the same circle from a common point are ≅

Formulas, Facts and Tips(cont'd)

MORE CIRCLE FACTS



A parallelogram inscribed in a circle is a rhombus A circle circumscribed



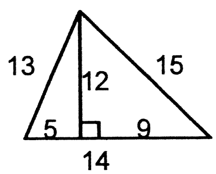
A quadrilateral can be inscribed in a circle if the opposite angles are supplementary

For diagonals x, y: xy = ac + bd

The area of the quadrilateral is $A=\sqrt{s(s-a)(s-b)(s-c)(s-d)}$ where s = ½ (a + b + c + d)

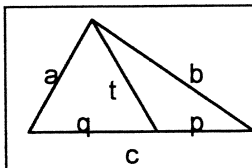
MORE OBSCURE FACTS:

13-14-15 Δ



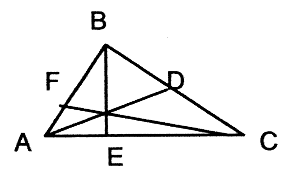
Composed of two rt. Δs that share a common leg:
5-12-13 rt. Δ and 9-12-15 rt. Δ

Stewart's Thm



$$a^2p+b^2q=c(t^2+pq)$$

≅ Segments in ≅ Δ s



$$\frac{AE}{EC}=\frac{CD}{BD}=\frac{BF}{AF}=1$$

Euler's Formula

F= # faces, E= # edges, V= # vertices: F − E + V = 2

CONICS

Circles radius r, center (h, k) $x^2+y^2=r^2$ $(x-h)^2+(y-k)^2=r^2$	**Parabola** $y=ax^2+bx+c$ Axis of symmetry: $x=-b/2a$ If $a<0$, opens down and T.P. is a maximum Sideways when $x=ay^2+by+c$; if $a<0$, opens left Given: vertex (h, k) focus $(h, k+p)$ Axis of symmetry $x=h$ directrix $y=k-p$ Then $y-k=\frac{1}{4p}(x-h)^2$
Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ When $a>b$, ellipse is horizontal. Length of major axis $=2a$, minor axis length $=2b$ Eccentricity $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}$ where c = distance from center to a focus	
Hyperbola $xy=k$ *or* Inverse variation Asymptotes are coordinate axes	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Hyperbola with horizontal transverse axis of length 2a (vertex to vertex) Conjugate axis length $=2b$, where $a^2+b^2=c^2$ and c = distance from center to focus $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ has a vertical transverse axis

NYSML PRACTICE NOTES: ADVANCED CONCEPTS AND FORMULAS

FACTORING

Difference of two fourths: $(a^4 - b^4) = (a - b)(a^3 + a^2b + ab^2 + b^3)$

Difference of two n^{th} powers: $(a^n - b^n) = (a - b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1})$

*Sum of two n^{th} powers, where n is **odd** and n > 0*

$$(a^n + b^n) = (a + b)(a^{n-1} - a^{n-2}b + ... - ab^{n-2} + b^{n-1})$$

POLYNOMIAL EQUATIONS, ROOTS AND COEFFICIENTS

Quadratic: $(x-a)(x-b) = 0$ *where roots are a, b*

$$x^2 - (a+b)x + ab = 0$$

Cubic: $(x-a)(x-b)(x-c) = 0$ *where roots are a, b, c*

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0$$

Fourth Degree: $(x-a)(x-b)(x-c)(x-d) = 0$ *where roots are a, b, c, d*

$$x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - (abd+acd+bcd+abc)x + abcd = 0$$

Notice the terms whose coefficients are the **opposite of the <u>sum of the roots</u>** and the **<u>product of the roots</u>**, and also how the other coefficients relate to the roots.

Polynomial function: $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, where $a_i \in Reals$

The **zeros** or **roots** of a function are all those values of x where $f(x) = 0$.

<u>Factor Theorem:</u> If $x = k$ is a zero of $f(x)$ then $(x - k)$ is a factor of $f(x)$.

Given the polynomial equation $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ then

The *sum of the roots* is $\frac{-a_1}{a_0}$ and the *product of the roots* is $\frac{(-1)^n a_n}{a_0}$

SYNTHETIC DIVISION can be used TO

1. Divide a polynomial by a binomial (x + ***a***) and determine remainders if any.
   *Use divisor value d = **-a***
2. Determine if a given binomial (***x- r***) is a factor of a polynomial expression
   and/or
   Determine if a value ***r*** is a root of a polynomial equation.
   *Use divisor value d = **r**. Remainder must be 0 to be a factor or root.*
3. Evaluate a polynomial f(x) for a given value of the variable, ***a***
   *Use divisor value d = **a** and **f(a)** will be the **remainder**.*

MODULAR ARITHMETIC, Residue Or Remainder Systems

These systems consist of a finite number of elements or classes of integers, where a class consists of all integers that have the same remainder.

X mod n ≡ remainder when X/n

<u>Modular Addition:</u> Mod(sum) ≡ Mod(sum(mods))

<u>Modular Multiplication:</u> Mod(product) ≡ Mod(product(mods))

<u>Fermat's Theorem</u>: If ***p*** is prime and ***a*** is relatively prime to ***p*** then

$$a^{p-1} \equiv 1 \pmod{p}$$

Used in cyclic patterns, for example equivalent powers of i $(i^n = i^{n \bmod 4})$
and also in questions involving powers of numbers and units digits.

Note: If m is prime, then all the numbers 0, 1, 2,...m-1 form a field under the operations of modular addition and multiplication.

LOGARITHMIC FUNCTION

The log function base a is the name given to the inverse of the exponential function base a, thus

$$a^{\log_a x} = x \quad \text{and} \quad \log_a(a^x) = x$$

NUMBERS USING BASES OTHER THAN 10

1. *To change from base n to base 10 use*

$$(abc)_n = a(n)^2 + b(n)^1 + c(n)^0$$

2. *To change a base 10 number to base n*

   Divide the number by the base and your remainder will be the units digit.
   Now divide your dividend by the base to get the next higher place.
   Continue this process until your dividend is zero.
   Note: ***Every digit should be less n***

For bases greater than 10 (like in hex decimal), letters are used for digit values over 9 as follows:
A = 10, B = 11, C = 12, D = 13, E = 14, F = 15

EXAMPLES: $10101_2 = 1(2)^4 + 0(2)^3 + 1(2)^2 + 0(2)^1 + 1(2)^0 = 21$

The base ten number 31 $= 1101_3$ and $25 = 1F_{16}$

DIVISOR NOTATION **(d | n)**

If a number ***n*** is divided by the divisor ***d***, a quotient ***q*** will be obtained with remainder ***r*** such that $n = dq + r$, where $r < d$

If $r = 0$, then we say that ***d divides n*** or **(d | n)**

INFINITE REPLACEMENT

In general, look for the ***pattern of repetition*** and ***make an appropriate substitution*** so that the ***infinite nature of the expression is replaced with some closed form.***

Examples: To find *x*, when

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \dots}}}$$

use $x = 3 + \frac{1}{x}$

To find x, when $2 = x^{x^{x^{\cdots}}}$ use $2 = x^2$

MEANS

Arithmetic mean (average) of ***n*** numbers is the ***sum*** of the numbers divided by ***n***

Geometric mean of *n* numbers is the ***n^{th} root*** of the ***product*** of the numbers.

Harmonic mean of *n* numbers, $a_1, a_2, a_3, \dots a_n$ is the reciprocal of the arithmetic mean of each of the reciprocals.

$$HM = n\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} \right)^{-1}$$

Example if n = 2 then $HM = 2\left(\frac{a_1 a_2}{a_1 + a_2} \right)$

Note: Harmonic mean $\leq$ geometric mean $\leq$ arithmetic mean

ABSOLUTE VALUE FUNCTION

There are two alternate definitions for absolute value.

$$ABS\ (x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

$$ABS\ (x) = |x| = \sqrt{x^2}$$

PYTHAGOREAN TRIPLES

A Pythagorean triple is ***primitive*** if all three numbers are relatively prime. Example: 3, 4, 5

If p and q are ***relatively prime***, then $(p^2 - q^2, 2pq, p^2 + q^2)$ is a Pythagorean triple.

If n is ***odd,*** then $\left(n, \frac{1}{2}(n^2 - 1), \frac{1}{2}(n^2 + 1)\right)$ is a Pythagorean triple.